

On domain decomposition preconditioners for finite element approximations of the Helmholtz equation using absorption

IVAN GRAHAM, UNIVERSITY OF BATH, UK

This is joint work with Paul Childs (Schlumberger Gould Research, Cambridge), Martin Gander (Geneva), Euan Spence (Bath) and Douglas Shanks (Bath).

As a model problem for high-frequency wave scattering, we study the boundary value problem

$$\left. \begin{aligned} -(\Delta + k^2)u &= f && \text{in } \Omega, \\ \frac{\partial u}{\partial n} - iku &= g && \text{on } \Gamma, \end{aligned} \right\} \quad (1)$$

where Ω is a bounded domain in $\mathbb{R}^d$ with boundary Γ . Our results also apply to sound-soft scattering problems in truncated exterior domains. Finite element approximations of this problem for high wavenumber k are notoriously hard to solve. The analysis of Krylov space-based iterative solvers such as GMRES is also hard, since the corresponding system matrices are complex, non-Hermitian and usually highly non-normal and so information about spectra and condition numbers of the system matrices generally does not give much information about the convergence rate of iterative methods.

Quite a lot of recent research has focussed on preconditioning (1) using an approximate solution of the “shifted Laplace” problem

$$\left. \begin{aligned} -(\Delta + k^2 + i\epsilon)u &= f && \text{in } \Omega, \\ \frac{\partial u}{\partial n} - i\mu(k, \epsilon)u &= g && \text{on } \Gamma, \end{aligned} \right\} \quad (2)$$

for some function μ .

Let A, A_ϵ denote the system matrices for discretizations of (1) and (2) respectively, and let B_ϵ^{-1} denote any (practically useful) approximate inverse for A_ϵ . It is easy to see that sufficient conditions for B_ϵ^{-1} to be a good GMRES preconditioner for A are: (i) A_ϵ^{-1} should be a good preconditioner for A and (ii) B_ϵ^{-1} should be a good preconditioner for A_ϵ . It is generally observed that (i) holds if the “absorption” parameter $\epsilon > 0$ is not taken too large, while (ii) holds (e.g. for geometric multigrid) provided ϵ is large enough. However there is no rigorous explanation for these observations.

The first part of the talk will explore sufficient conditions on ϵ so that (i) holds. This analysis uses techniques from PDE analysis of (1) and (2) in the high frequency case.

In the second part of the talk we consider condition (ii), analysing the case when B_ϵ^{-1} is defined by classical additive Schwarz domain decomposition methods. The analysis here is quite different to the classical analysis of Cai and Widlund, which depends on k being sufficiently small. Here we use a coercivity argument in the natural k -dependent energy norm to estimate the field of values of the preconditioned matrix. This analysis holds for k arbitrarily large.

The analysis shows that there is a gap between the ranges of ϵ which ensure conditions (i) and (ii). Practical exploration of the performance of the solver in the gap suggests that efficient algorithms can still be constructed for solving this problem with high wavenumber k using several variants of classical domain decomposition methods. New directions for future analysis are also suggested by the experiments.